# 3.3 Investigate Properties of Quadrilaterals

Rectangular shapes dominate the skylines of most cities in North America. This section investigates the properties of rectangles and other types of quadrilaterals.

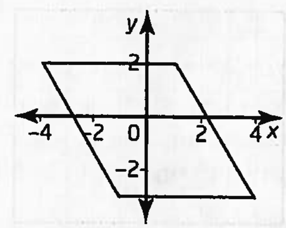
## Investigate

**Tools**
- grid paper
- ruler
- compasses
- protractor

### What properties does a parallelogram have?

**Method 1: Use Pencil and Paper**

1. Draw a large parallelogram on grid paper. Describe your method and explain how you know that it produces a parallelogram.




2. Draw the diagonals of the parallelogram. Describe the location of the point where the diagonals intersect.

3. Make a conjecture about how the point of intersection of the diagonals divides each diagonal.

4. Fold your parallelogram drawing to test your conjecture. Describe your findings. Compare your results with those of several classmates.

5. Reflect What can you conclude about the diagonals of the parallelograms drawn by you and your classmates? Do you think that this conclusion applies to all parallelograms? Explain your reasoning.

6. Mark the midpoint of each side of the parallelogram. Join the midpoints of adjacent sides to form a quadrilateral within the parallelogram. What properties does the new quadrilateral appear to have?

7. Make a conjecture about the type of quadrilateral formed by joining the midpoints of adjacent sides of a parallelogram.

8. Use measurements or paper-folding to determine whether your conjecture is true. Describe your findings, and compare them with those of several classmates.

9. Reflect What can you conclude about the midpoints of the sides of the parallelograms drawn by you and your classmates? Do you think that this conclusion applies to all parallelograms? Explain.

**Method 2: Use *The Geometer's Sketchpad®***

**Tools**

- computer with *The Geometer's Sketchpad®*

1. Turn on the grid display and automatic labelling of points.

2. Construct line segment AB and point C above it. Connect B to C with a line segment. Select point C and line segment AB. Choose **Parallel Line** from the **Construct** menu. Select point A and line segment BC. Then, choose **Parallel Line** from the **Construct** menu again.

3. Select the two lines that you constructed, and choose **Intersection** from the **Construct** menu. Select the two lines again, and choose **Hide Parallel Lines** from the **Display** menu.

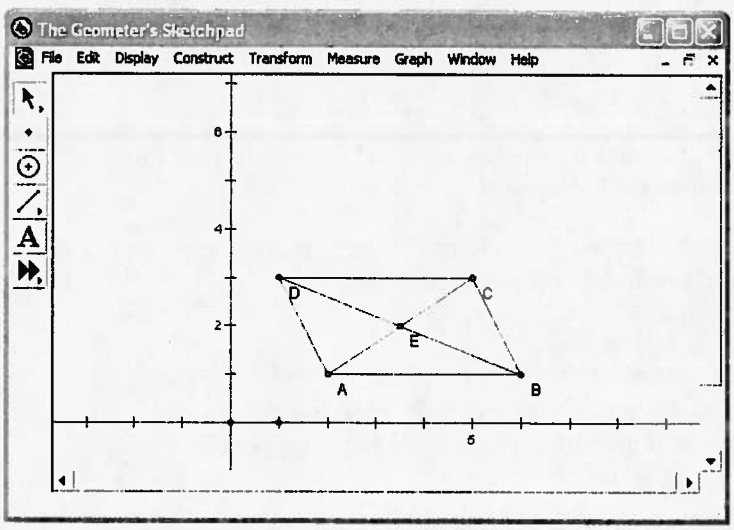
**Technology Tip**

The keyboard shortcut for the **Hide** option is Ctrl+H.

4. Construct line segments from C to D and from D to A.

5. Explain how you know that quadrilateral ABCD is a parallelogram.

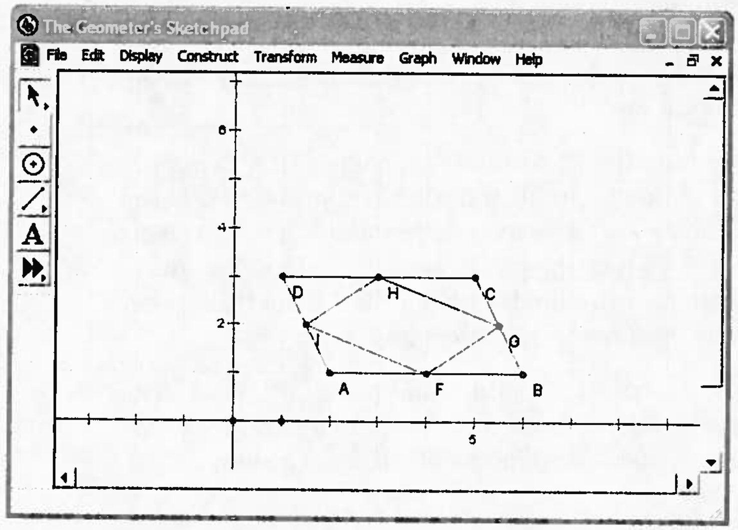
6. Construct diagonals AC and BD. Select the two diagonals and choose **Intersection** from the **Construct** menu. Measure the distance from the intersection point E to each of the four vertices. What can you conclude from these measurements?




7. Drag each of the vertices of ABCD to various new locations. Does ABCD remain a parallelogram? What do you notice about the lengths of AE, BE, CE, and DE?

8. Reflect What property do the diagonals of parallelograms have? Explain your reasoning.

9. Hide or delete the diagonals and their measurements. Construct the midpoint of each side of the parallelogram. Construct line segments to join the midpoints of adjacent sides. What properties does the quadrilateral formed by these line segments appear to have?




10. Make a conjecture about the type of quadrilateral formed by joining the midpoints of adjacent sides of a parallelogram.

11. Use measurements to determine whether your conjecture is true. Describe your findings.

12. Observe your measurements of quadrilateral FGHI as you drag the vertices of ABCD around the screen. Do any of the relationships among the measurements change?

13. Reflect What can you conclude about the midpoints of the sides of a parallelogram? Explain how you know that this conclusion applies to all parallelograms.

**Tools**

- TI-83 Plus or TI-84 Plus graphing calculator

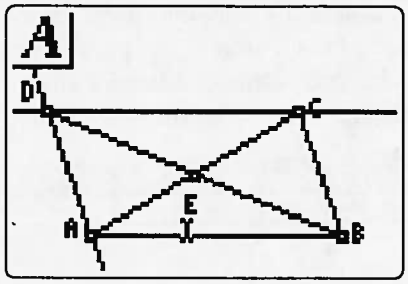
**Method 3: Use a Graphing Calculator**

1. Start the Cabri® Jr. application. Use the **F5** menu to show the axes if they do not appear on the screen. Choose **Segment** from the **F2** menu, and draw a line segment AB near the bottom of the screen. Draw another segment from B to a point C above the segment AB. Choose **Alpha-Num** from the **F5** menu. Label the points A, B, and C.

2. Choose **Parallel** from the **F3** menu. Move the cursor to segment AB until it flashes, and press ENTER. Select point C, segment BC, and point A in the same way.

3. Choose **Point** from the **F2** menu; then, choose **Intersection** from the submenu. Move the cursor to the line through point A and press ENTER. Then, move the cursor to the line through point C and press ENTER again. Label the intersection as point D.

4. Choose **Segment** from the **F2** menu, and construct diagonal line segments from A to C and from B to D. Then, choose **Intersection** from the **F2** menu, and construct the intersection of the line segments. Label this intersection as point E.



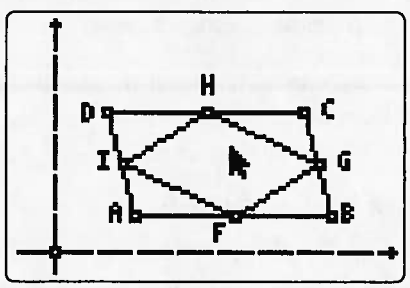

5. Choose **Measure** from the **F5** menu; then, choose **D. & Length.** Move the cursor to point A until it flashes and press ENTER. Select point E in the same way. Move the measurement to a convenient location. Press ENTER to lock the measurement in place. Use the same procedure to measure the lengths of BE, CE, and DE. What can you conclude from these measurements?

6. Press CLEAR. Move the cursor to point B and press ALPHA. Drag vertex B to various new locations. Does ABCD remain a parallelogram? What do you notice about the lengths of AE, BE, CE, and DE?

7. Reflect What property do the diagonals of parallelograms have? Explain your reasoning.

8. Press ALPHA. Choose **Hide/Show** from the **F5** menu; then, choose **Object**. Hide the diagonals and their measurements. Choose **Midpoint** from the **F3** menu. Select point A and then point B. Select point B again; then, select point C. Use the same procedure to construct the midpoints of CD and AD. Label the midpoints F, G, H, and I.

9. Construct line segments to join the midpoints of adjacent sides. What properties does the quadrilateral formed by these line segments appear to have?




10. Make a conjecture about the type of quadrilateral formed by joining the midpoints of adjacent sides of a parallelogram.

11. Use measurements to determine whether your conjecture is true. Describe your findings.

12. Observe your measurements of quadrilateral FGHI as you drag vertex B around the screen. Do any of the relationships among the measurements change?

13. Reflect What can you conclude about the midpoints of the sides of a parallelogram? Explain how you know that this conclusion applies to all parallelograms.

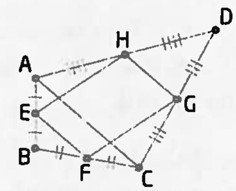
## Example 1 Midpoints of a Quadrilateral

Draw any quadrilateral ABCD and find the midpoint of each side. Form a smaller quadrilateral EFGH inside the original one by drawing line segments joining the midpoints of adjacent sides. Investigate the properties of the smaller quadrilateral.

### Solution

The inner quadrilateral EFGH appears to be a parallelogram. To test this conjecture, investigate the properties of the sides of EFGH.

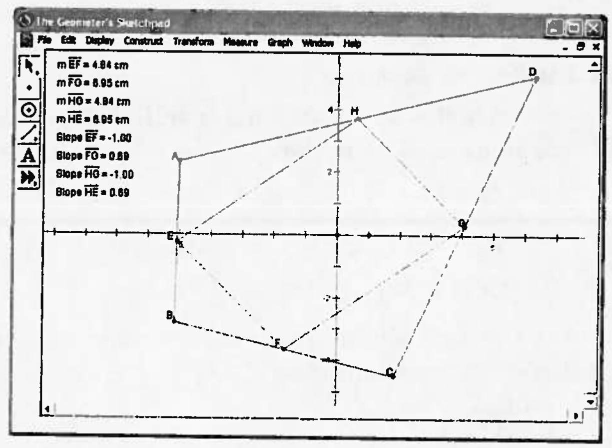
**Method 1: Use Pencil and Paper**




Draw the diagonal from vertex A to vertex C. In △DAC, line segment GH joins the midpoints of sides AD and CD. Any line segment joining the midpoints of two sides of a triangle is parallel to the third side. Therefore, GH is parallel to CA. Similarly, EF joins the midpoints of two sides of △ABC, so EF is also parallel to CA. Therefore, EF is parallel to GH.

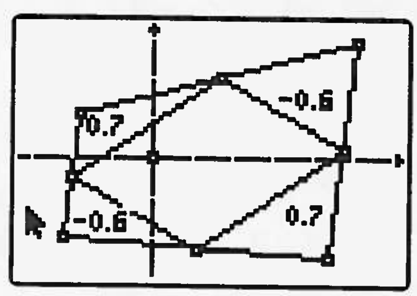
Applying the same properties in △ABD and △BCD shows that EH is parallel to FG. Since both pairs of opposite sides in quadrilateral EFGH are parallel, it is a parallelogram.

**Method 2: Use Geometry Software**




Construct any quadrilateral. Then, construct the midpoint of each side. Add line segments joining the midpoints of adjacent sides.

Measure the slope of each side of the inner quadrilateral. The measurements show that the opposite sides are parallel.




When the vertices of the original quadrilateral are dragged around the screen, the slopes of the opposite sides remain equal.

Therefore, the quadrilateral formed by joining the midpoints of adjacent sides of any quadrilateral is a parallelogram.

**Did You Know?**

The parallelogram formed by connecting the midpoints of adjacent sides of a quadrilateral is called a **Varignon parallelogram** after the French mathematician Pierre Varignon (1654–1722).

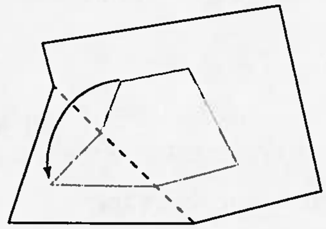
## Example 2 Midpoints of a Trapezoid

Investigate the properties of the midpoints of the non-parallel sides of a trapezoid.


### Solution

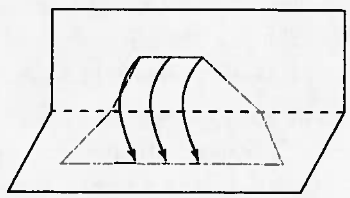
**Method 1: Use Paper Folding**

Draw a large trapezoid on a sheet of paper. Find the midpoint of each non-parallel side by folding your drawing so that one end is on top of the other.



Then, draw a line segment joining the midpoints of the non-parallel sides. This line segment appears to be parallel to the parallel sides of the trapezoid.

Now, fold the trapezoid along the line segment joining the midpoints of the non-parallel sides. The parallel sides of the trapezoid line up perfectly.



Therefore, the line segment joining the midpoints of the non-parallel sides is parallel to the other two sides and lies halfway between them.

**Method 2: Use Analytic Geometry**

Draw run-rise triangles on the two non-parallel sides. The sum of the runs of these sides is equal to the difference between the lengths of the parallel sides. So,

$QR = PS + run_{PQ} + run_{RS}$

Similarly, $TU = PS + run_{PT} + run_{US}$.

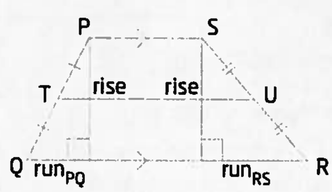
The run from the midpoint of a side to either endpoint is equal to half the run between the endpoints. So,

$$TU = PS + \frac{run_{QP}}{2} + \frac{run_{RS}}{2}$$

$$= \frac{2PS + run_{QP} + run_{RS}}{2}$$

$$= \frac{PS + (PS + run_{QP} + run_{RS})}{2}$$

$$= \frac{PS + QS}{2}$$




The length of the line segment joining the midpoints of the non-parallel sides is equal to the mean of the lengths of the parallel sides.